A 278-vertex construction for $n = 70$. Let B_m denote the book consisting of m triangles sharing a common edge. We construct a graph G on

$$N = 4n - 2 = 278$$

vertices such that

$$B_{69} \not\subseteq G \qquad \text{and} \qquad B_{70} \not\subseteq \overline{G}.$$

Set

$$q = 2n - 1 = 139, \qquad \ell = \frac{q-1}{2} = 69,$$

and let

$$\chi : \mathbb{F}_{139} \longrightarrow \{-1, 0, 1\}$$

be the quadratic character, extended by $\chi(0) = 0$. Since

$$139 \equiv 3 \pmod 8,$$

we have

$$\chi(-1) = \chi(2) = -1.$$

The element

$$\rho = 2$$

is a primitive root modulo 139. Indeed, since $138 = 2 \cdot 3 \cdot 23$,

$$2^{69} \equiv -1, \qquad 2^{46} \equiv 96, \qquad 2^{6} \equiv 64 \pmod{139},$$

so the order of 2 is 138. Hence

$$g = \rho^2 = 4$$

has multiplicative order 69. Thus

$$H = \langle g \rangle$$

is the subgroup of nonzero squares in $\mathbb{F}_{139}$, and

$$\mathbb{F}_{139}^{\times} = H \sqcup (-H).$$

Define binary sequences

$$a = (a_t)_{t \in \mathbb{Z}_{69}}, \qquad b = (b_t)_{t \in \mathbb{Z}_{69}}$$

by

$$a_0 = 1, \qquad a_t = \chi(g^t - 1) \quad (t \neq 0),$$

and

$$b_t = -\chi(g^t + 1) \quad (t \in \mathbb{Z}_{69}).$$

All exponents and sequence indices are understood modulo 69. Because $-1 \notin H$, one has $g^t + 1 \neq 0$ for every t, so both a and b take values in $\{-1, 1\}$.

For complete numerical reproducibility, their $+1$-supports are

$$\{t : a_t = 1\} = \{0, 3, 4, 6, 7, 8, 11, 12, 13, 14, 15, 17, 18, 19, 21, 25, 28, 30, 31, 33, 35, 37, \\ 40, 42, 43, 45, 46, 47, 49, 53, 59, 60, 64, 67, 68\},$$

and

$$\{t : b_t = 1\} = \{0, 2, 8, 9, 11, 12, 13, 17, 18, 19, 20, 23, 25, 28, 31, 32, 33, 34, 35, 36, 37, \\ 38, 41, 44, 46, 49, 50, 51, 52, 56, 57, 58, 60, 61, 67\}.$$

We record the character identities needed below. We use the standard quadratic-character sum

$$\sum_{x \in \mathbb{F}_q} \chi(Ax^2 + Bx + C) = -\chi(A) \tag{1}$$

whenever $A \neq 0$ and $B^2 - 4AC \neq 0$. For completeness, after completing the square, this reduces to

$$\sum_{y \in \mathbb{F}_q} \chi(y^2 - \delta) = -1 \qquad (\delta \neq 0).$$

Indeed, the number of pairs (y,z) satisfying

$$z^2 = y^2 - \delta$$

is both

$$\sum_y \bigl(1 + \chi(y^2 - \delta)\bigr)$$

and $q-1$, since

$$(y-z)(y+z) = \delta$$

has exactly one solution (y,z) for each choice of $y - z \in \mathbb{F}_q^\times$.

First,

$$\begin{aligned}
\sum_{h\in H} \chi(h-1) &= \frac{1}{2} \sum_{x\in\mathbb{F}_{139}^\times} (1+\chi(x))\chi(x-1) \\
&= \frac{1}{2}\left(\sum_{x\neq 0} \chi(x-1) + \sum_{x\neq 0} \chi(x(x-1))\right) \\
&= \frac{1}{2}(1-1) = 0.
\end{aligned}$$

The first sum is 1, since the omitted $x = 0$ term in the complete linear-character sum is $\chi(-1) = -1$, and the second sum is -1 by (1). Therefore

$$\sum_{t\in\mathbb{Z}_{69}} a_t = 1 + \sum_{h\in H} \chi(h-1) = 1.$$

Similarly,

$$\begin{aligned}
\sum_{h\in H} \chi(h+1) &= \frac{1}{2} \sum_{x\in\mathbb{F}_{139}^\times} (1+\chi(x))\chi(x+1) \\
&= \frac{1}{2}\left(\sum_{x\neq 0} \chi(x+1) + \sum_{x\neq 0} \chi(x(x+1))\right) \\
&= \frac{1}{2}(-1-1) = -1,
\end{aligned}$$

and hence

$$\sum_{t\in\mathbb{Z}_{69}} b_t = -\sum_{h\in H} \chi(h+1) = 1.$$

The sequences also satisfy

$$b_0 = -\chi(2) = 1.$$

For every t,

$$\begin{aligned}
b_{-t} &= -\chi(g^{-t}+1) \\
&= -\chi\left(\frac{1+g^t}{g^t}\right) \\
&= -\chi(1+g^t) = b_t,
\end{aligned}$$

because g^t is a square. For $t \neq 0$,

$$\begin{aligned}
a_{-t} &= \chi(g^{-t}-1) \\
&= \chi\left(\frac{1-g^t}{g^t}\right) \\
&= \chi(-1)\chi(g^t-1) \\
&= -a_t.
\end{aligned}$$

Thus

$$b_{-t} = b_t, \qquad a_{-t} = -a_t \quad (t \neq 0). \tag{2}$$

For a binary sequence c, write

$$\mathrm{PAF}_c(s) = \sum_{t\in\mathbb{Z}_{69}} c_t c_{t+s}.$$

We next prove that

$$\mathrm{PAF}_a(s) + \mathrm{PAF}_b(s) = -2 \qquad (s \neq 0). \tag{3}$$

Fix $s \neq 0$ and put

$$r = g^s \in H \setminus \{1\}.$$

If

$$\alpha_t = \chi(g^t - 1),$$

then $a_t = \alpha_t + \mathbf{1}_{\{t=0\}}$. Expanding the autocorrelation gives

$$\mathrm{PAF}_a(s) = \sum_t \alpha_t \alpha_{t+s} + \alpha_s + \alpha_{-s}.$$

By (2), the last two terms cancel, and hence

$$\mathrm{PAF}_a(s) = \sum_{h\in H} \chi\big((h-1)(rh-1)\big). \tag{4}$$

For b, no exceptional zero occurs, so

$$\mathrm{PAF}_b(s) = \sum_{h\in H} \chi\big((h+1)(rh+1)\big).$$

Substituting $x = -h$ gives

$$\mathrm{PAF}_b(s) = \sum_{x\in -H} \chi\big((x-1)(rx-1)\big). \tag{5}$$

Since $H \sqcup (-H) = \mathbb{F}_{139}^{\times}$, equations (4) and (5) imply

$$\mathrm{PAF}_a(s) + \mathrm{PAF}_b(s) = \sum_{x\in\mathbb{F}_{139}^{\times}} \chi\big((x-1)(rx-1)\big).$$

The polynomial

$$(x-1)(rx-1) = rx^2 - (r+1)x + 1$$

has leading coefficient r, which is a square, and discriminant

$$(r+1)^2 - 4r = (r-1)^2 \neq 0.$$

Therefore, by (1),

$$\sum_{x\in\mathbb{F}_{139}} \chi\big((x-1)(rx-1)\big) = -1.$$

The $x = 0$ term is 1, so deleting it gives

$$\mathrm{PAF}_a(s) + \mathrm{PAF}_b(s) = -2,$$

proving (3).

Let X and Y be the 69×69 circulant matrices

$$X_{rs} = a_{s-r}, \qquad Y_{rs} = b_{s-r}.$$

Let

$$I = I_{69}, \qquad J = J_{69}, \qquad \mathbf{j} = (1, \ldots, 1) \in \mathbb{R}^{1\times 69}.$$

The sequence sums give

$$XJ = JX = YJ = JY = J. \tag{6}$$

The symmetry relations (2) give

$$Y^T = Y, \qquad X + X^T = 2I. \tag{7}$$

Moreover, X, X^T, Y, and J are all circulant matrices, and therefore commute with one another. We shall also use

$$J^2 = 69J. \tag{8}$$

The diagonal entries of $XX^T + Y^2$ are

$$69 + 69 = 138.$$

For distinct indices, equation (3) shows that the corresponding entry is -2. Hence

$$XX^T + Y^2 = 140I - 2J. \tag{9}$$

Order the 278 vertices as

$$u,\ v,\ A_0 \times \mathbb{Z}_{69},\ A_1 \times \mathbb{Z}_{69},\ B_0 \times \mathbb{Z}_{69},\ B_1 \times \mathbb{Z}_{69}.$$

Define the following 278×278 Seidel matrix:

$$S = \begin{pmatrix} 0 & -1 & \mathbf{j} & \mathbf{j} & -\mathbf{j} & -\mathbf{j} \\ -1 & 0 & -\mathbf{j} & \mathbf{j} & \mathbf{j} & -\mathbf{j} \\ \mathbf{j}^T & -\mathbf{j}^T & Y - I & -Y & -X & X \\ \mathbf{j}^T & \mathbf{j}^T & -Y & -Y + I & -X & -X \\ -\mathbf{j}^T & \mathbf{j}^T & -X^T & -X^T & Y - I & Y \\ -\mathbf{j}^T & -\mathbf{j}^T & X^T & -X^T & Y & -Y + I \end{pmatrix}. \tag{10}$$

Because $a_0 = b_0 = 1$, the diagonal of S is zero. All its off-diagonal entries belong to $\{-1, 1\}$, and S is symmetric.

Define the graph G by

$$xy \in E(G) \iff S_{xy} = 1.$$

Equivalently,

$$A_G = \frac{S + J_{278} - I_{278}}{2}. \tag{11}$$

Using (6)–(9), together with the commutativity of the circulant blocks, direct block multiplication gives

$$S\mathbf{1}_{278} = -\mathbf{1}_{278} \tag{12}$$

and

$$S^2 = \begin{pmatrix} 277 & 0 & 0 & 0 & -4\mathbf{j} & 0 \\ 0 & 277 & -4\mathbf{j} & 0 & 0 & 0 \\ 0 & -4\mathbf{j}^T & 281I - 2Y - 2J & 0 & 2X - 2J & 0 \\ 0 & 0 & 0 & 281I - 2Y - 2J & 0 & -2X - 2J \\ -4\mathbf{j}^T & 0 & 2X^T - 2J & 0 & 281I - 2Y - 2J & 0 \\ 0 & 0 & 0 & -2X^T - 2J & 0 & 281I - 2Y - 2J \end{pmatrix}. \tag{13}$$

For example, the identities $X + X^T = 2I$ and $XY = YX$ are needed for the vanishing off-diagonal blocks in this multiplication.

Since every entry of X and Y is ± 1, every off-diagonal entry of S^2 is either 0 or -4. Indeed, off the diagonal,

$$281I - 2Y - 2J = -2(Y + J),$$

while

$$2X - 2J = 2(X - J), \qquad -2X - 2J = -2(X + J),$$

and similarly for the transposed blocks. Consequently,

$$S^2 = 277I_{278} - 4K \tag{14}$$

for a symmetric 0/1-matrix K with zero diagonal. In particular,

$$(S^2)_{xy} \le 0 \qquad (x \ne y). \tag{15}$$

Equation (12) shows that G is regular. Indeed, if $d(x)$ denotes the degree of x, then

$$(S\mathbf{1})_x = d(x) - \big(277 - d(x)\big) = 2d(x) - 277.$$

Thus

$$2d(x) - 277 = -1,$$

and hence

$$d(x) = 138 \qquad \text{for every } x \in V(G). \tag{16}$$

Now let $x \ne y$ and suppose first that $xy \in E(G)$. Let $c_G(x, y)$ be the number of common neighbours of x and y in G. Among the other 276 vertices, there are

$$c_G(x, y)$$

vertices adjacent to both x and y,

$$2\big(137 - c_G(x, y)\big)$$

vertices adjacent to exactly one of them, and

$$c_G(x, y) + 2$$

vertices adjacent to neither. Therefore

$$\begin{aligned} (S^2)_{xy} &= c_G(x, y) + \big(c_G(x, y) + 2\big) - 2\big(137 - c_G(x, y)\big) \\ &= 4\big(c_G(x, y) - 68\big). \end{aligned} \tag{17}$$

By (15),

$$c_G(x, y) \le 68.$$

Thus no edge of G lies in 69 triangles, and consequently

$$B_{69} \nsubseteq G. \tag{18}$$

Suppose instead that $xy \notin E(G)$. Then xy is an edge of $\overline{G}$. Since G is 138-regular, its complement is 139-regular. Let

$$c_{\overline{G}}(x, y)$$

be the number of common neighbours of x and y in $\overline{G}$. Applying the same signed count to this edge of the 139-regular complement gives

$$(S^2)_{xy} = 4\big(c_{\overline{G}}(x, y) - 69\big). \tag{19}$$

Again using (15), we obtain

$$c_{\overline{G}}(x, y) \le 69.$$

Therefore no edge of $\overline{G}$ lies in 70 triangles, and hence

$$B_{70} \nsubseteq \overline{G}. \tag{20}$$

We have constructed a graph on 278 vertices satisfying

$$B_{69} \nsubseteq G \qquad \text{and} \qquad B_{70} \nsubseteq \overline{G}.$$

It follows that

$$R(B_{69}, B_{70}) > 278.$$

Together with the general upper bound

$$R(B_{n-1}, B_n) \le 4n - 1,$$

applied at $n = 70$, this gives

$$R(B_{69}, B_{70}) \le 279.$$

Therefore

$$\boxed{R(B_{69}, B_{70}) = 279}.$$